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Simple systematics in the energy eigenvalues of quantum anharmonic oscillators

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Abstract

In an earlier paper we proposed (Bhattacharya R, Roy D and Bhowmick S 1998 *Phys. Lett.* A **244** 9) a remarkably simple scheme for evaluating the ground state energy of λx^{2m} quantum anharmonic oscillators. In the present paper we extend the scheme to evaluate the excited state energies as well, and provide a motivation for the scheme. The only information needed in the calculation is the first term of the standard strong coupling expansion for each state of the oscillators. We have calculated the first few terms of the standard strong coupling expansion for some excited states for the quartic, sextic and octic oscillators and, for a given m, we provide simple expressions for the strong coupling constants as functions of n, which seem to reproduce any excited state energy with a reasonable accuracy. Moreover, for the ground state, we propose a simple expression for the first term of the strong coupling expansion which is globally true, i.e., it gives the value of the constant for any m. We then calculate a large number of excited state energies for the quartic, sextic and octic oscillators over a wide range of values for the coupling parameter. For any of the states the problem reduces to one of solving a polynomial equation and the predicted values closely agree with those obtained by other methods.

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1. Introduction

One of the basic problems in non-relativistic quantum mechanics is to find the energy eigenvalues of a microsystem with an appropriate potential. There is no procedure for dealing systematically with even the apparently simpler problems in one-dimensional quantum mechanics, such as the anharmonic oscillator (AHO). In recent years the problem of AHO's has been a very active area of research. The AHO's serve as a rather simple model for understanding the non-trivial features of a large number of complicated systems and have played a very significant role in the theoretical understanding of many branches of physics,

namely, nuclear and particle physics, atomic and molecular physics and condensed matter physics. Moreover, it serves as a basis for checking different approximate methods in quantum mechanics, the simplified counterpart of field-theoretical models, etc.

The pioneering works of Bender and Wu [1] and Simon [2] have generated a vast amount of literature on AHO's. The article by Killingbeck [3] provides an overview of some of the earlier work and more recent references can be found in [4–10]. Though seemingly simple, it is not an easy problem to find the energy spectrum and eigenfunctions of the AHO's. The standard way of solving this problem is to invoke perturbation theory. However, the application of the conventional perturbation method gives a divergent perturbation series for all values of the anharmonic coupling constant even for a quartic anharmonic oscillator [1], and although a renormalized series [5] improves summability significantly, the series stays divergent. Padé approximants are able to sum the perturbation series for the quartic anharmonic oscillator for small values of the coupling parameter [2, 11, 12]. For the sextic oscillator, although it is guaranteed that the $[n + j, n]$ Padé approximants converge to a unique limit for every $j \geqslant -1$ as $n \to \infty$, the convergence is so slow that the method is not computationally useful [2, 12]. For the octic anharmonic oscillator, the perturbation coefficients grow much more rapidly and the Padé approximants are not able to sum the series [13, 14]. Recently other nonlinear sequence transforms have been used with considerable success for the summation of these strongly divergent perturbation series [4, 5, 15]. However, these summation methods, as well, are not effective for large coupling constants. On the other hand, one can build a strong coupling expansion and this works quite well for large values of the coupling constant [16, 17]. It is possible to apply summation techniques that use simultaneously the information from the renormalized weak coupling as well as the strong coupling expansion [18, 19]. Apart from the perturbation expansion coupled with different summation methods, several other methods have been applied to find the energy spectrum of the AHO's. These include the variational method [20–22], the Hill determinant [23, 24] and the Riccati-Hill determinant method [25], the hypervirial perturbation expansion [26], the WKB methods [27, 28] and even this list is far from exhaustive.

The vast literature on AHO's indicates the difficulty in obtaining the complete energy spectrum as well as the importance of the problem. In an earlier paper [29] we proposed a remarkably simple scheme for obtaining the ground state energies of the quantum anharmonic oscillators. The problem reduces to that of solving a simple polynomial equation. In the present paper, we extend the scheme for obtaining the complete energy spectrum of the oscillators. As in the case of the ground state energy, the excited state energies for the λx^{2m} oscillator defined by the Hamiltonian $H = p^2 + x^2 + \lambda x^{2m}$ are also given by a polynomial equation of the same degree and the predicted energy values closely agree with those predicted by other methods and is given by

$$\left(\frac{E^{(m,n)}}{2n+1}\right)^{(m+1)} - \left(\frac{E^{(m,n)}}{2n+1}\right)^{(m-1)} = \left(K_0^{(m,n)}\right)^{(m+1)}\lambda, \tag{1}$$

where $E^{(m,n)}$ is the nth excited state energy of the λx^{2m} oscillator and $K_0^{(m,n)}$ are constants. Thus for a particular oscillator, the energy eigenvalues for a given state are expressed in terms of a single constant for any value of λ. It is seen trivially that at $\lambda = 0$ the above equation requires $E^{(m,n)} = 2n + 1$, the result for the harmonic oscillator. Also, for $m = 1$, equation (1) reduces to $E^{(1,n)}(\lambda) = (2n + 1)\sqrt{1 + \lambda}$, with $K_0^{(1,n)} = 1$, as it should. The dependence of $K_0^{(m,n)}$ on n for a given m and that of $K_0^{(m,0)}$ on m will be discussed in a subsequent section.

The purpose of the present paper is not merely to obtain accurate energy eigenvalues of different oscillators for which a large number of methods exist in the literature. Rather, we

propose a simple scheme which provides the energy eigenvalues with a reasonable accuracy almost without any effort. In the next section, we propose a justification for such a simple scheme. In the subsequent section we shall show how simply modifying the constant $K^{(m,n)}$ on the right-hand side of (19) to fit the results for large λ makes it quiet accurate over the whole range of λ. We also discuss how the energy spectrum for the different oscillators obtained from this simple formula compares with the values obtained by other methods.

2. Motivation for the scheme

The Schrödinger equation for the $2m$th order anharmonic oscillators, in a suitably chosen set of units, is

$$H^{(m)}(\lambda)\psi = -\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + (x^2 + \lambda x^{2m})\psi = E\psi, \tag{2}$$

where λ is the strength of the anharmonic term. The values of $m = 2, 3, 4, 5\ldots$ correspond respectively to the quartic, sextic, octic and decadic oscillators and so on. For $\lambda = 0$, this is essentially the harmonic oscillator equation and we know that the solutions are Hermite polynomials times $\exp(-x^2/2)$. On the other hand, the large distance behaviour of the wavefunction is dominated by the λx^{2m} term and it is easy to see that as $x \to \pm\infty$, ψ goes like $\exp(-\sqrt{\lambda}|x|^{(m+1)}/(m+1))$. An *ansatz* for the wavefunction, inspired by the one used by Ginsberg and Montroll [30], that captures both the small and large distance behaviour of ψ is

$$\psi(x) = F(x)\psi_0(x), \tag{3}$$

where $F(x)$ is a polynomial in x and $\psi_0(x)$ is given by

$$\psi_0(x) = \exp(f(x)) = \exp\left(-\sqrt{ax^4 + \sum_{i=1}^{m-1}\beta_i x^{2i+4}}\right). \tag{4}$$

Here we must have $\beta_{m-1} = \lambda/(m+1)^2$, while we know that $a \to 1/4$ as $\lambda \to 0$. In order to reproduce the correct harmonic oscillator result, the function $F(x)$ must approach the Hermite polynomial as $\lambda \to 0$. We shall assume that all the coefficients $\beta_1, \beta_2, \ldots, \beta_{m-2}$ of the intermediate powers are small, and that we need only to retain them to first order in the subsequent calculations.

Using the smallness of the β_i's we now expand $f(x) = \ln\psi_0(x)$ in a power series in x and retaining only the terms that are linear in β_i's we have

$$\begin{aligned} f(x) &= -\sqrt{a}x^2\left(1 + \sum_{i=1}^{m-1}\frac{\beta_i}{a}x^{2i}\right)^{1/2} \\ &\simeq -\sqrt{a}x^2\left(1 + \frac{1}{2}\sum_{i=1}^{m-1}\frac{\beta_i}{a}x^{2i}\right). \end{aligned} \tag{5}$$

The two derivatives of $f(x)$, which we will need in a short while, are

$$f'(x) = -\sqrt{a}\left(2x + \sum_{i=1}^{m-1}(i+1)\frac{\beta_i}{a}x^{2i+1}\right) \tag{6}$$

$$f''(x) = -\sqrt{a}\left(2 + \sum_{i=1}^{m-1}(i+1)(2i+1)\frac{\beta_i}{a}x^{2i}\right). \tag{7}$$

Substituting equation (3) into equation (2) we get

$$F''(x) + P(x)F'(x) + Q(x)F(x) = 0, \tag{8}$$

where

$$\begin{aligned} P(x) &= 2\frac{\psi_0'(x)}{\psi_0(x)} = 2f'(x) \\ &= -2\left(2\sqrt{a}x + \sum_{i=1}^{m-1}(i+1)\frac{\beta_i}{\sqrt{a}}x^{2i+1}\right) \end{aligned} \tag{9}$$

and

$$\begin{aligned} Q(x) &= \frac{\psi_0''(x)}{\psi_0(x)} + (E - x^2 - \lambda x^{2m}) \\ &= f''(x) + f'(x)^2 + E - x^2 - \lambda x^{2m} \\ &\approx (E - 2\sqrt{a}) + \left(4a - 1 - \frac{6\beta_1}{\sqrt{a}}\right)x^2 \\ &\quad + \sum_{i=2}^{m-1}\left(4i\beta_{i-1} - (i+1)(2i+1)\frac{\beta_i}{\sqrt{a}}\right)x^{2i} + \mathcal{O}(x^{2m}). \end{aligned} \tag{10}$$

In the above we have ignored terms that are quadratic in β_i for $i = 1, \ldots, m-2$. Also, we have retained terms up to x^{2m-1} (the effect of the λx^{2m} term already being accounted for in the large distance behaviour of the wavefunction). This leads us to drop terms like $\beta_i\beta_{m-1}$ which are only of the first order in the small β_is (β_{m-1} is not small unless λ is). Of course, this means that we should not expect very good results from this approach at large values of λ.

Now we assume that $F(x)$ is a polynomial in x of degree n. Substituting $F(x) = \sum_{i=0}^{n} c_i x^i$ into equation (8) and equating the coefficients of $x^n, x^{n+2}, \ldots, x^{n+2m-2}$ to zero yields the following set of equations:

$$E - 2\sqrt{a}(2n+1) = 0 \tag{11}$$

$$4a - 1 - \frac{6\beta_1}{\sqrt{a}}\left(1 + \frac{2n}{3}\right) = 0. \tag{12}$$

For oscillators beyond the quartic ($m > 2$), for $i = 2, 3, \ldots, m-1$ we have

$$4i\beta_{i-1} - (i+1)(2i+1+2n)\frac{\beta_i}{\sqrt{a}} = 0. \tag{13}$$

For the quartic oscillator, $\beta_1 = \lambda/9$ and we can eliminate a from equations (11) and (12) to get the following simple equation for the energy eigenvalues:

$$\left(\frac{E^{(2,n)}}{2n+1}\right)^3 - \left(\frac{E^{(2,n)}}{2n+1}\right) = \frac{4}{3}\left(1 + \frac{2n}{3}\right)\lambda. \tag{14}$$

Ginsberg and Montroll [30] had derived the same equation for the ground state and first excited state of the quartic oscillator by following a similar approach. It will perhaps not be amiss here to mention that for the second excited state, the approach of Ginsberg and Montroll yields a more complicated equation, whose asymptotic solution for large λ is

$$E^{(2,2)\prime} \sim 4.844\lambda'^{1/3} + 0.409\,\lambda'^{-1/3} + \cdots \tag{15}$$

while our simple equation leads to (after correcting for the scale differences, ($E^{(2,2)\prime} = E^{(2,2)}/2$ and $\lambda' = \lambda/2$)

$$E^{(2,2)\prime} \sim 4.598\lambda'^{1/3} + 0.453\,\lambda'^{-1/3} + \cdots \tag{16}$$

These are to be compared with the exact asymptotic form [31]

$$E^{(2,2)\prime} \sim 4.697\lambda'^{1/3} + 0.494\,\lambda'^{-1/3} + \cdots \tag{17}$$

It should be clear that the simpler equation (equation (14)) yields results that are as good as, if not better, than the complicated approach.

Of course, we expect (14) to be more accurate for small values of λ rather than large values. For example, the energy eigenvalue it yields for the first excited state deviates from the exact value by roughly 0.5% for $\lambda = 0.2$, while the error is ∼3% for $\lambda = 2000$. The error is also expected to grow for higher excited states. For the seventh excited state, the above errors are ∼6% and ∼8%, respectively.

For higher order oscillators ($m > 2$), we solve equation (13) for β_1 to get

$$\begin{aligned}\beta_1 &= \beta_{m-1}\prod_{i=2}^{m-1}\frac{(i+1)(2i+1+2n)}{i4\sqrt{a}} \\ &= \frac{m}{2^{m-1}(m+1)^2}\left(\prod_{i=2}^{m-1}(2i+1+2n)\right)\left(\frac{E^{(m,n)}}{2n+1}\right)^{-(m-1)}\lambda,\end{aligned} \tag{18}$$

which leads to the equation

$$\left(\frac{E^{(m,n)}}{(2n+1)}\right)^{m+1} - \left(\frac{E^{(m,n)}}{(2n+1)}\right)^{m-1} = (K^{(m,n)})^{(m+1)}\lambda, \tag{19}$$

where

$$K^{(m,n)} = \left[\frac{m}{2^{m-3}(m+1)^2}\left(\prod_{i=1}^{m-1}(2i+1+2n)\right)\right]^{1/(m+1)}, \tag{20}$$

where $m = 3, 4, \ldots, n = 1, 2, \ldots$. We have already seen that for a quartic oscillator

$$K^{(2,n)} = \sqrt[3]{\frac{4}{3}\left(1+\frac{2n}{3}\right)} \tag{21}$$

which is a special case of equation (20) for $m = 2$. Hence equation (20) may be considered as a general equation which is valid for all $m = 2, 3, 4, \ldots$.

The simple form given by equation (19) follows from the crucial approximation where we have ignored all higher powers of the intermediate coefficients $\beta_1, \beta_2, \ldots, \beta_{m-2}$. This is a good approximation for small λ and consequently this is the regime where equation (19) is expected to work well. Since there are no intermediate coefficients for the quartic oscillator, we expect the equation to work better for the quartic oscillator than the higher order oscillators.

We should emphasize at this point that the derivation of the approximate equation (19) shows clearly that this equation will only be accurate for small λ. As we will show in the next section, we can modify the equation by replacing $K^{(m,n)}$ by a different constant $K_0^{(m,n)}$—which will allow us to get a better fit with the exact results over the whole range of λ.

3. The scheme for calculation of the energy spectrum

In section 2 we arrived at an equation (equation (19)) which is identical to equation (1), except for the fact that $K_0^{(m,n)}$ is replaced by $K^{(m,n)}$. Equation (19) gives fair agreement for the energy eigenvalues for small λ and small n. For large λ, the energy eigenvalues calculated from (19) deviates significantly from the exact values, as is only to be expected from the nature of the

approximations involved. The precise form of $K_0^{(m,n)}$ in equation (1) depends on the different approximation schemes that can be adopted. One interesting feature of equation (1) is the fact that if we change the value of $K_0^{(m,n)}$, its effect is increasingly important for large λ. For a specific example, let us consider the quartic oscillator. If we change the value of $K_0^{(2,0)}$ from 1.2 to 1.5, the energy eigenvalue of the quartic oscillator at $\lambda = 0.2$ changes by less than 2%, whereas that for $\lambda = 2000$ changes by about 8%. It is thus evident that a reasonably fair agreement for the energy eigenvalues can be obtained over entire range of λ by choosing $K_0^{(m,n)}$ so that the energy eigenvalues for large λ are well reproduced.

The Rayleigh–Schrödinger perturbation series for the energy of the anharmonic oscillators, given by

$$E^{(m,n)}(\lambda) = \sum_{j=0}^{\infty} b_j^{(m,n)} \lambda^j \tag{22}$$

is known to diverge. Here, $n = 0$ corresponds to the ground state. The coefficients $b_j^{(m,n)}$, for m and n, grow rapidly with j and the radius of convergence of the perturbation series is strictly zero. A variety of summation methods exist for extracting finite numbers out of this series for small λ, but these are mostly concerned with the ground state energy. There is no straightforward summation method for large λ [32].

It is possible to calculate $E^{(m,n)}(\lambda)$ using a strong coupling expansion [2]:

$$E^{(m,n)}(\lambda) = \lambda^{1/(m+1)} \sum_{j=0}^{\infty} K_j^{(m,n)} \lambda^{-2j/(m+1)}, \tag{23}$$

which is obtained by transforming the Hamiltonian into an equivalent one using Symanzik scaling. Weniger [16] calculated $K_j^{(m,0)}$ directly from $b_j^{(m,0)}$ for $m = 2$, 3 and 4. For large λ, the first term in equation (23) is the most important term and we shall identify $K^{(m,n)}$ in equation (19) with $K_0^{(m,n)}/(2n+1)$, $K_0^{(m,n)}$ being the first term of the strong coupling expansion. It may be noted that for large λ, $(E^{(m,n)})^{m+1}$ is proportional to λ. Thus, the values of $K_0^{(m,n)}$ for different m and n can be estimated by evaluating the energy eigenvalues for some large value of λ ($\lambda = 10^6$) and using the relation

$$E^{(m,n)}(\lambda) \cong \lambda^{1/(m+1)} K_0^{(m,n)}. \tag{24}$$

We now estimate the energy values for large λ by utilizing the representation [33]

$$\psi(x) = \sum_i a_i(E) x^i R_\gamma(x) \tag{25}$$

with an appropriate reference function $R_\gamma(x)$. The coefficients $a_i(E)$ depend on E and are readily generated by substituting equation (25) into the Schrödinger equation for the anharmonic oscillator (equation (2)). It can be shown [33] that the convergent zeros of the coefficient function

$$a_i\left(E_l^{(i)}\right) = 0 \tag{26}$$

(where l labels the roots) converge to the exact energy eigenvalues, i.e.,

$$\lim_{i\to\infty} E_l^{(i)} = E_l^{(\text{exact})}. \tag{27}$$

Using $R_\gamma(x) = \mathrm{e}^{-\gamma x^2}$ gives the recurrence relation

$$a_i(E) = \frac{((2\gamma - 3)2\gamma - E)a_{i-2} + (1 - 4\gamma^2)a_{i-4} + \lambda a_{i-2-2m}}{i(i-1)}. \tag{28}$$

Table 1. Values of $K_0^{(m,n)}$ for different m and n.

n	$K_0^{(2,n)}$	$K_0^{(3,n)}$	$K_0^{(4,n)}$
0	1.0604	1.1448	1.2258
1	1.2666	1.4516	1.5853
2	1.4911	1.8187	2.0627
3	1.6634	2.1370	2.4886
4	1.8068	2.4157	2.8771
5	1.9307	2.6663	3.2355

As i is increased, the convergence of the roots of equation (26) depends on the choice of γ. We adjust the value of γ so that all the roots of equation (26) are real and the $E_l^{(i)}$'s (for a given l) form a monotonically decreasing sequence for increasing i. We have evaluated the roots of equation (26) for $i = 60$ and have repeatedly applied the ϵ_2-transform [34, 35] on the sequence $E_l^{(i)}$ to extract the limit $i \to \infty$.

The values of $K_0^{(m,n)}$ for different m and n are shown in table 1. For the ground state $K_0^{(m,0)}$ has been evaluated by different authors [36, 37] for $m = 2, 3, 4, 5$. For the quartic oscillator, $K_0^{(2,n)}$ has been evaluated by Skăla *et al* [36] for $n = 1, 2, \ldots, 10$ and the values shown in table 1 are as reported by these authors. For the sextic and decadic oscillators the values of $K_0^{(m,n)}$ for the excited states have been estimated by evaluating the energy for a very large value of λ ($= 10^6$), by the method as outlined above, and using relation (24).

To obtain the n dependence of $K_0^{(m,n)}$, we note that the renormalization scheme [38] replaces $\lambda \in (0, \infty)$ by a renormalized coupling parameter $\kappa \in (0, 1)$:

$$\lambda = \frac{1}{B^{(m,n)}} \frac{\kappa}{(1-\kappa)^{m+1/2}}, \tag{29}$$

where

$$B^{(m,0)} = \frac{m(2m-1)!!}{2^{m-1}}$$

and the energy $E^{(m,n)}(\lambda)$ can be expressed as [4]

$$E^{(m,n)}(\lambda) = (1-\kappa)^{-1/2} E_R^{(m,n)}(\kappa). \tag{30}$$

It can be shown [5] that in the infinite limit

$$K_0^{(m,n)} = \lim_{t \to 0} \frac{E^{(m,n)}(\lambda)}{\lambda^{1/m+1}} = (B^{(m,n)})^{1/(m+1} E_R^{(m,n)}(1). \tag{31}$$

The variational condition leads to the following expressions for $B^{(m,n)}$ [39]:

$$\begin{aligned} B^{(2,n)} &= \frac{3(2n^2+2n+1)}{2n+1} \\ B^{(3,n)} &= \frac{15(4n^3+6n^2+8n+1)}{4(2n+1)}. \\ B^{(4,n)} &= \frac{35(2n^4+4n^3+10n^2+8n+3)}{4(2n+1)} \end{aligned} \tag{32}$$

which are rational functions in n of order $[m/1]$. In the simple approximation (equation (20)) $(K^{(m,n)})^{m+1}$ was a polynomial of order $m-1$. To obtain the n dependence of $K_0^{(m,n)}$ for a given m, we assume that $\left(K_0^{(m,n)}\right)^{m+1}$ can be represented by rational functions in n of order $[m/1]$.

Table 2. Values of $K_0^{(2,n)}$, for $n = 5, 6, \ldots, 10$, obtained from equation (33) are given in the first row. In the second row we give the values reported in [36].

n	5	6	7	8	9	10
$K_0^{(2,n)}$	1.9307	2.0406	2.1399	2.2308	2.3148	2.3931
	1.9291	2.0383	2.1369	2.2271	2.3106	2.3884

Table 3. Energy eigenvalues of quartic anharmonic oscillator. For each value of λ the values within brackets are the exact energy eigenvalues.

λ	0.2	10	100	2000
$E^{(2,0)}$	1.10 (1.12)	2.43 (2.45)	4.99 (5.00)	13.39 (13.39
$E^{(2,1)}$	3.49 (3.54)	8.56 (8.60)	17.81 (17.83)	47.94 (47.94)
$E^{(2,2)}$	6.20 (6.28)	16.58 (16.64)	35.85 (35.87)	94.03 (94.03)
$E^{(2,3)}$	9.14 (9.26)	25.74 (25.81)	54.35 (54.39)	146.82 (146.84)
$E^{(2,4)}$	12.29 (12.44)	35.79 (35.89)	75.81 (75.88)	204.93 (205.03)
$E^{(2,5)}$	15.60 (15.80)	46.60 (46.73)	98.91 (99.03)	267.52 (267.75)
$E^{(2,6)}$	19.08 (19.32)	58.08 (58.24)	123.45 (123.64)	334.03 (334.42)
$E^{(2,7)}$	22.70 (22.97)	70.14 (70.35)	149.28 (149.55)	404.03 (404.62)
$E^{(2,8)}$	26.45 (26.76)	82.75 (83.00)	176.28 (176.63)	477.22 (478.02)
$E^{(2,9)}$	30.32 (30.68)	95.85 (96.16)	204.36 (204.79)	553.33 (554.37)
$E^{(2,10)}$	34.30 (34.70)	109.42 (109.77)	233.44 (233.97)	632.16 (633.45)
$E^{(2,11)}$	38.40 (38.84)	123.42 (123.82)	263.45 (264.08)	713.53 (715.08)
$E^{(2,12)}$	42.59 (43.07)	137.82 (138.28)	294.34 (295.07)	797.29 (799.12)
$E^{(2,13)}$	46.88 (47.41)	152.61 (153.13)	326.06 (326.91)	883.32 (885.43)
$E^{(2,14)}$	51.26 (51.83)	167.76 (168.34)	358.58 (359.54)	971.49 (973.90)
$E^{(2,15)}$	56.34 (56.34)	183.26 (183.90)	391.85 (392.92)	1061.71 (1064.43)
$E^{(2,16)}$	60.94 (60.62)	199.10 (199.80)	425.84 (427.03)	1153.89 (1156.93)
$E^{(2,17)}$	64.93 (65.62)	215.25 (216.02)	460.52 (462.84)	1247.95 (1251.32)
$E^{(2,18)}$	69.65 (70.38)	231.72 (232.55)	495.86 (497.31)	1343.82 (1347.52)
$E^{(2,19)}$	74.44 (75.22)	248.48 (249.37)	531.85 (533.43)	1441.43 (1445.48)
$E^{(2,20)}$	79.31 (80.12)	265.52 (266.49)	568.46 (570.17)	1540.72 (1545.48)
$E^{(2,21)}$	84.24 (85.10)	282.85 (283.88)	605.67 (607.51)	1641.64 (1646.40)
$E^{(2,22)}$	89.25 (90.15)	300.44 (301.54)	643.45 (645.44)	1744.13 (1749.26)
$E^{(2,23)}$	94.32 (95.27)	318.29 (319.46)	681.80 (683.93)	1848.15 (1853.66)
$E^{(2,24)}$	99.46 (100.45)	336.40 (337.64)	720.70 (722.96)	1953.66 (1959.55)
$E^{(2,25)}$	104.67 (105.69)	354.75 (356.06)	760.12 (762.52)	2060.61 (2066.86)

From the values given in table 1 we make rational interpolations to obtain the n dependence of $K_0^{(m,n)}$ and these are given by

$$
\begin{aligned}
K_0^{(2,n)} &= \left(\frac{(1.1924 + 33.2383n + 56.2169n^2)}{1 + 43.6106n}\right)^{1/3} \\
K_0^{(3,n)} &= \left(\frac{(1.7176 + 1.3224n + 2.6933n^2 + 0.7092n^3)}{1 + 0.4510n}\right)^{1/4} \\
K_0^{(4,n)} &= \left(\frac{(2.7676 + 12.6576n + 9.2212n^2 + 13.3678n^3 + 6.4509n^4)}{1 + 3.1840n}\right)^{1/5}.
\end{aligned}
\tag{33}
$$

Table 4. Energy eigenvalues of sextic anharmonic oscillator. For each value of λ the values within brackets are the exact energy eigenvalues.

λ	0.2	10	100	2000
$E^{(3,0)}$	1.13 (1.17)	2.16 (2.21)	3.69 (3.72)	7.69 (7.70)
$E^{(3,1)}$	3.76 (3.90)	8.04 (8.11)	13.94 (13.95)	29.20 (29.12)
$E^{(3,2)}$	7.18 (7.38)	16.56 (16.64)	28.97 (28.98)	60.91 (60.81))
$E^{(3,3)}$	11.29 (11.55)	27.07 (27.16)	47.56 (47.56)	100.16 (100.04)
$E^{(3,4)}$	15.99 (16.30)	39.19 (39.29)	69.05 (69.05)	145.53 (145.39)
$E^{(3,5)}$	21.21 (21.56)	52.72 (52.85)	93.05 (93.07)	196.23 (196.14)
$E^{(3,6)}$	26.89 (27.29)	67.53 (67.70)	119.31 (119.40)	251.72 (251.75)
$E^{(3,7)}$	33.00 (33.46)	83.50 (83.73)	147.63 (147.84)	311.61 (311.83)
$E^{(3,8)}$	39.51 (40.03)	100.55 (100.87)	177.92 (178.24)	375.58 (376.06)
$E^{(3,9)}$	46.40 (46.99)	118.62 (119.04)	210.00 (210.49)	443.39 (444.22)
$E^{(3,10)}$	53.65 (54.31)	137.65 (138.19)	243.80 (244.48)	514.82 (516.06)
$E^{(3,11)}$	61.23 (61.96)	157.60 (158.22)	279.23 (280.05)	589.71 (591.24)
$E^{(3,12)}$	69.15 (69.93)	178.42 (179.14)	316.21 (317.18)	667.90 (669.72)
$E^{(3,13)}$	77.37 (78.22)	200.09 (200.87)	354.70 (355.76)	749.25 (751.25)
$E^{(3,14)}$	85.90 (86.78)	222.56 (223.35)	394.62 (395.69)	833.64 (835.64)
$E^{(3,15)}$	94.72 (95.62)	245.81 (246.54)	435.93 (436.86)	920.98 (922.67)
$E^{(3,16)}$	103.82 (104.7)	269.82 (270.40)	478.59 (479.22)	1011.15 (1012.20)
$E^{(3,17)}$	113.19 (114.02)	294.56 (294.86)	522.55 (522.67)	1104.08 (1104.04)
$E^{(3,18)}$	122.83 (123.56)	320.02 (320.35)	567.77 (567.84)	1199.69 (1200.96)
$E^{(3,19)}$	132.72 (132.97)	346.16 (347.70)	614.22 (615.90)	1297.91 (1300.39)

To find the rational interpolant for $K_0^{(2,n)}$ we have used the four values for $n = 1, 2, 3$ and 4. To see how this interpolant reproduces the values for other values of n, we calculate $K_0^{(2,n)}$ for $n = 5, 6, \ldots, 10$ using the interpolant and compare with the values given in [36]. The agreement is fairly good and this gives us some confidence in the expressions for $K_0^{(3,n)}$ and $K_0^{(4,n)}$ as can be seen from table 2. Farnandez and Guardiola [37] have calculated $K_0^{(m,0)}$ for different m including $K_0^{(\infty,0)}$ and a *global* Padé approximant, which is valid for all m, given by

$$K_0^{(m,0)} = \frac{1 + 0.7961(m-1) - 0.051\,29(m-1)^2 - 0.046\,94(m-1)^3}{1 + 0.7858(m-1) - 0.1655(m-1)^2 + 0.019\,02(m-1)^3}. \quad (34)$$

To see how this approximant reproduces the values for different m, we calculate the values of $K_0^{(m,0)}$ for $m = 250$ and $m = 500$ from the above expression. These are respectively given by 2.3954 and 2.4307 and compares well with the values 2.400 235 and 2.433 557 reported by Farnandez and Guardiola [37].

We are now in a position to calculate the energy spectrum for the quartic, sextic and octic oscillators using equations (1) and (33) and these are displayed in tables 3–5 respectively. We have calculated 25 excited states for the quartic oscillator and 19 excited states for the sextic oscillator for different values of λ. As can be seen from tables 3 and 4, the agreement of these energy values with those reported by Meißner and Steinborn [39] is fairly good, the maximum discrepancy being of the order of 2% to 3% in some stray cases. In most of the cases, our simple formula reproduces the energy values with an accuracy of within 1%. We also mention that the energy eigenvalues for $\lambda = 400$, though not included in tables 3 and 4, are close to those reported in [39] except for $E^{(3,19)}$. Our calculated value for $E^{(3,19)}$ is 868.20, whereas the value reported in [39] is 834.02 and we presume that this is due to printing error. We suspect that the value should be 864.02.

Table 5. Energy eigenvalues of octic anharmonic oscillator. For a given n, the first row gives the values obtained from equation (1). The second row corresponds to the values obtained from equation (36) and the values within brackets give the exact energy eigenvalues.

λ	0.2	10	100	400
$E^{(4,0)}$	1.16	2.05	3.15	4.11
	1.21 (1.24)	2.10 (2.11)	3.18 (3.19)	4.14 (4.15)
$E^{(4,1)}$	4.08	7.87	12.24	16.07
	4.24 (4.28)	7.98 (7.93)	12.31 (12.20)	16.07 (15.95)
$E^{(4,2)}$	8.20	16.66	26.10	34.33
	8.46 (8.45)	16.80 (16.71)	26.20 (26.03)	34.40 (34.18)
$E^{(4,3)}$	13.45	27.97	43.98	57.91
	13.78 (13.45)	28.14 (28.02)	44.09 (43.90)	57.99 (57.74)
$E^{(4,4)}$	19.67	41.44	65.29	86.01
	20.05 (19.99)	41.63 (41.49)	65.42 (65.20)	86.11 (85.83)
$E^{(4,5)}$	26.77	56.84	89.67	118.17
	27.19 (27.12)	57.05 (56.90)	89.81 (89.57)	118.27 (117.96)
$E^{(4,6)}$	34.67	74.02	116.87	154.04
	35.14 (35.06)	74.24 (74.08)	117.01 (116.76)	154.15 (153.83)
$E^{(4,7)}$	43.33	92.85	146.70	193.39
	43.83 (43.73)	93.10 (92.93)	146.85 (146.60)	193.51 (193.18)
$E^{(4,8)}$	52.70	113.15	179.01	236.02
	53.23 (53.15)	113.51 (113.35)	179.17 (178.92)	236.14 (235.82)
$E^{(4,9)}$	62.75	135.15	213.68	281.77
	63.31 (63.23)	135.41 (135.26)	213.85 (213.62)	281.89 (281.59)
$E^{(4,10)}$	73.45	158.46	250.61	330.48
	74.04 (73.95)	158.74 (158.60)	250.78 (250.58)	330.62 (330.34)
$E^{(4,11)}$	85.31	183.14	289.71	382.06
	85.38 (85.31)	183.43 (183.31)	289.89 (289.71)	382.20 (381.97)
$E^{(4,12)}$	96.70	209.14	330.89	436.40
	97.33 (97.26)	209.44 (209.34)	331.08 (330.94)	436.54 (436.37)
$E^{(4,13)}$	109.20	236.41	374.10	493.41
	109.85 (109.80)	236.72 (236.64)	374.29 (374.18)	493.55 (493.14)
$E^{(4,14)}$	122.27	264.92	419.27	553.00
	122.94 (122.89)	265.24 (265.17)	419.46 (419.37)	553.15 (553.03)
$E^{(4,15)}$	135.88	294.63	466.34	615.11
	136.58 (136.49)	294.96 (295.13)	466.54 (466.93)	615.26

The energy eigenvalues for the first 15 excited states for the octic oscillator calculated form equation (1) and (33), along with the ground state, are shown in table 5. For this case the agreement worsens, the maximum disagreement being of the order of 7%. We note that, in deriving equation (19), we assumed that β_1, β_2 in equation (4) are small. If we relax this assumption then, for the octic oscillator, we will have all odd powes $E^{(4,n)}/(2n+1)$ in equation (19) with the restriction that the sum of the coefficients is zero. Keeping this in mind, we slightly modify the equation for the octic ocillator in the following form:

$$\left(\frac{E^{(4,n)}}{2n+1}\right)^5 - \frac{3}{2}\left(\frac{E^{(4,n)}}{2n+1}\right)^3 + \frac{1}{2}\left(\frac{E^{(4,n)}}{2n+1}\right) = \left(K_0^{(4,n)}\right)^5 \lambda. \tag{35}$$

For each $E^{(4,n)}$, the first row in table 5 gives the energy values obtained from equation (1) and in the row below we give the values obtained from equation (35) along with those

reported by Meißner and Steinborn [39]. It is seen from the table, that the values predicted by equation (35) are fairly close to the values reported in [39].

It has been brought to our notice that a recent paper by Matamala and Maldonado [40] calculates the energy levels of the quartic oscillator by using a modification of the well-known ladder operator approach. Their calculation is similar to the current one for the quartic oscillator in that it also involves the solution of a cubic equation. However, the cubic equation (14) (or the more accurate equation (1) for $m = 2$) directly leads to the energy eigenvalue, while the energy is a rational function of the root of the cubic equation in [40]. Moreover, the current approach easily generalizes to higher order anharmonic oscillators.

4. Conclusion

In conclusion, we reiterate the observation that a simple polynomial equation is sufficient to generate all energy levels of x^{2m} anharmonic oscillators with a fair degree of accuracy. A slight modification renders an even better agreement. The simplicity in the systematics of the energy levels suggests again a tantalizing structure in the actual, so far unknown, solution of the problem.

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